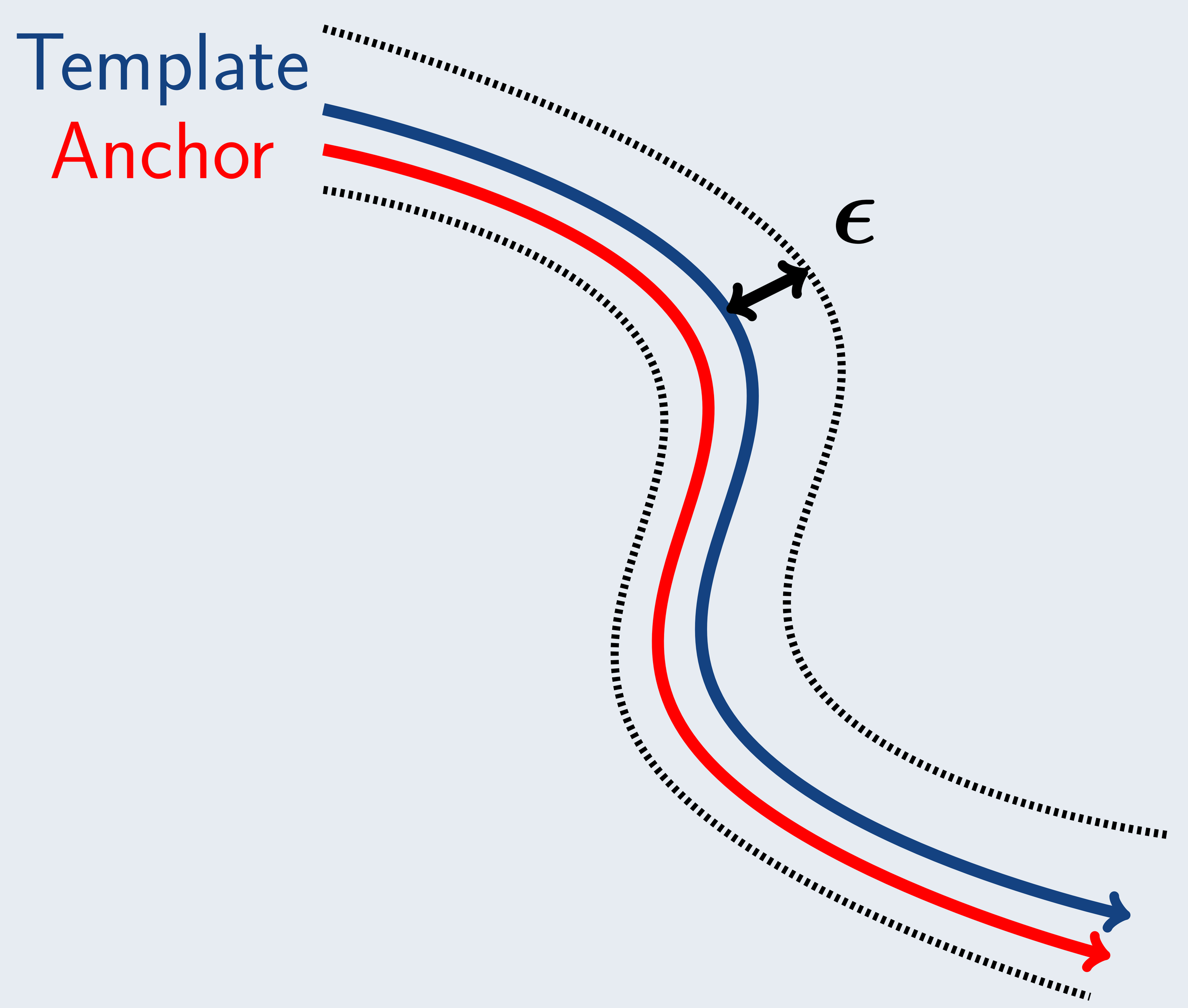
Formal Connections Between Template and Anchor Models

Vince Kurtz, Rafael Rodrigues da Silva, Patrick M. Wensing, and Hai Lin

What is Approximate Simulation?

Ensures that the anchor (whole-body) model can **track the template with ϵ precision**:



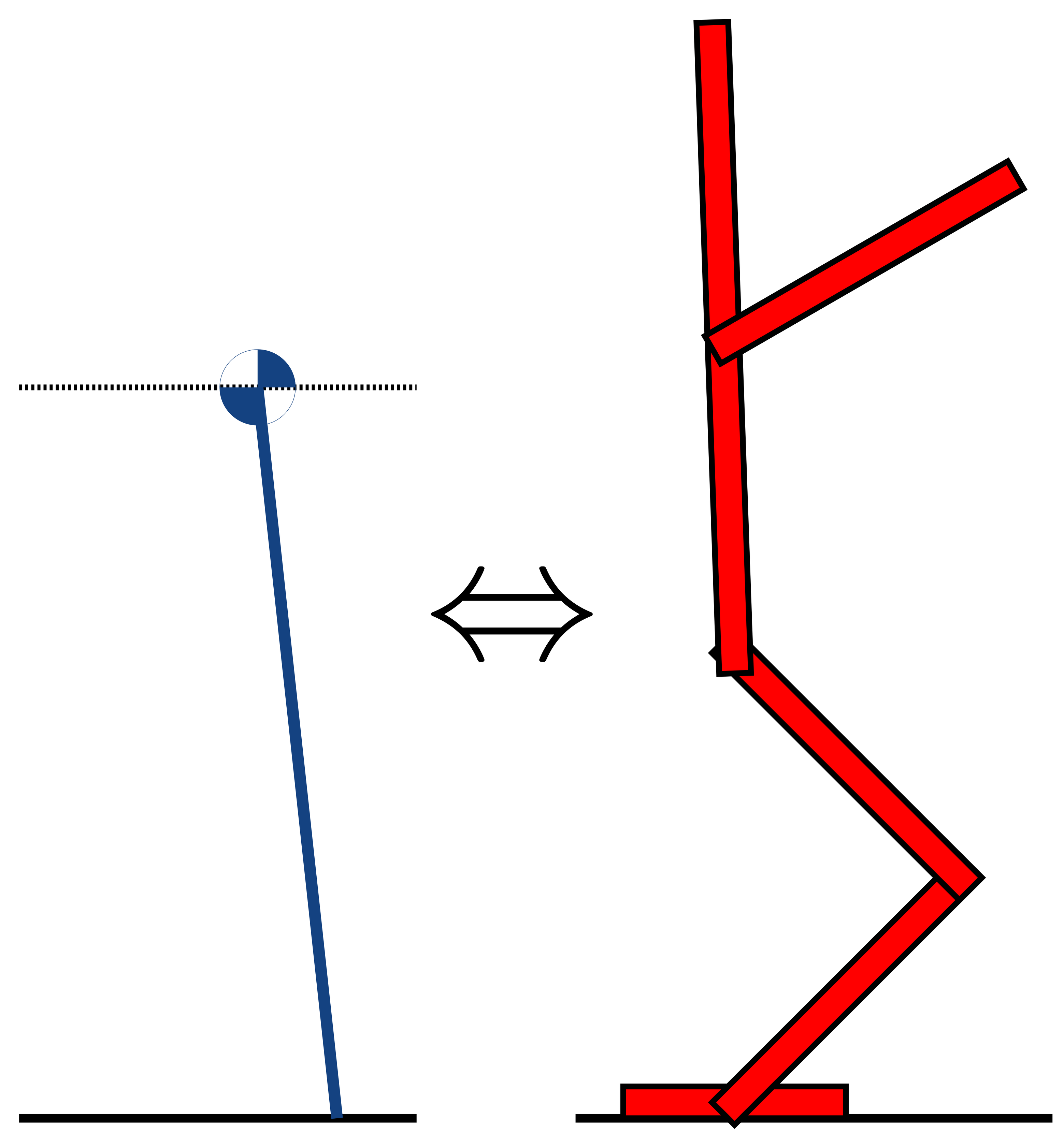
Certified with a **Lyapunov-like simulation function**:

$$\mathcal{V}(\mathbf{x}_1, \mathbf{x}_2) = \sqrt{(\mathbf{x}_1 - \mathbf{P}\mathbf{x}_2)^T \mathbf{M} (\mathbf{x}_1 - \mathbf{P}\mathbf{x}_2)}$$

Interface determines controls that track the template.

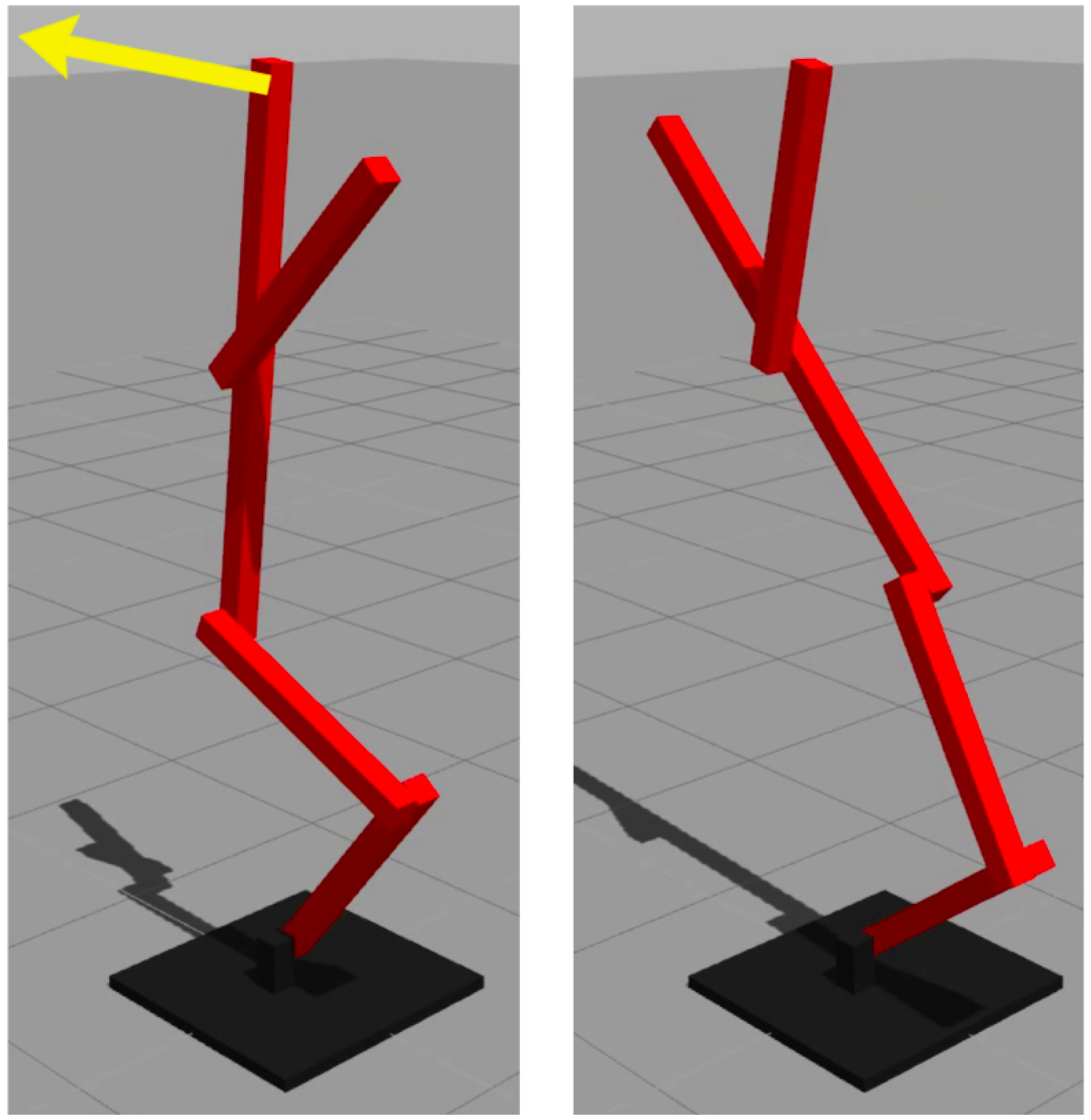
$$\mathbf{u}_1 = \mathbf{R}\mathbf{u}_2 + \mathbf{Q}\mathbf{x}_2 + \mathbf{K}(\mathbf{x}_1 - \mathbf{P}\mathbf{x}_2),$$

We establish an **Approximate Simulation Relation** between the **Linear Inverted Pendulum** and a **Planar Balancer**.



Template: Linear Inverted Pendulum

Anchor: Fully Actuated Balancer



Our approach enables **better push recovery** by allowing centroidal momentum to vary.

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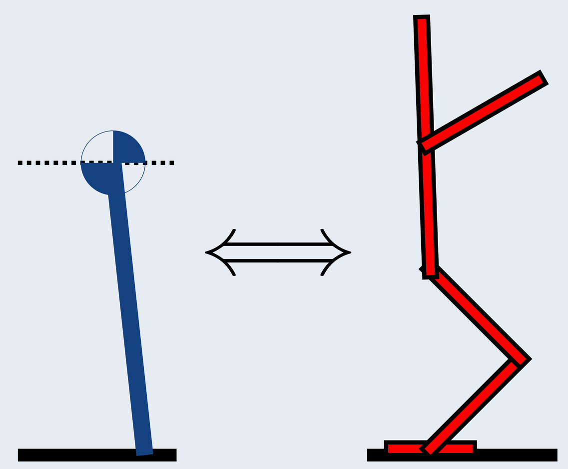

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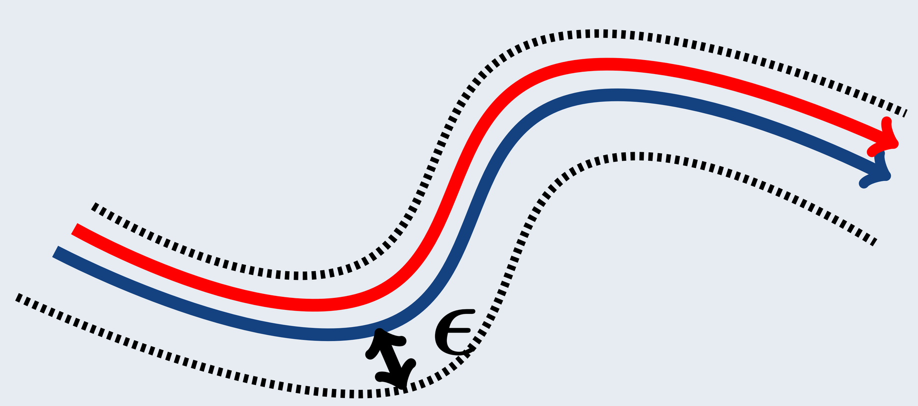
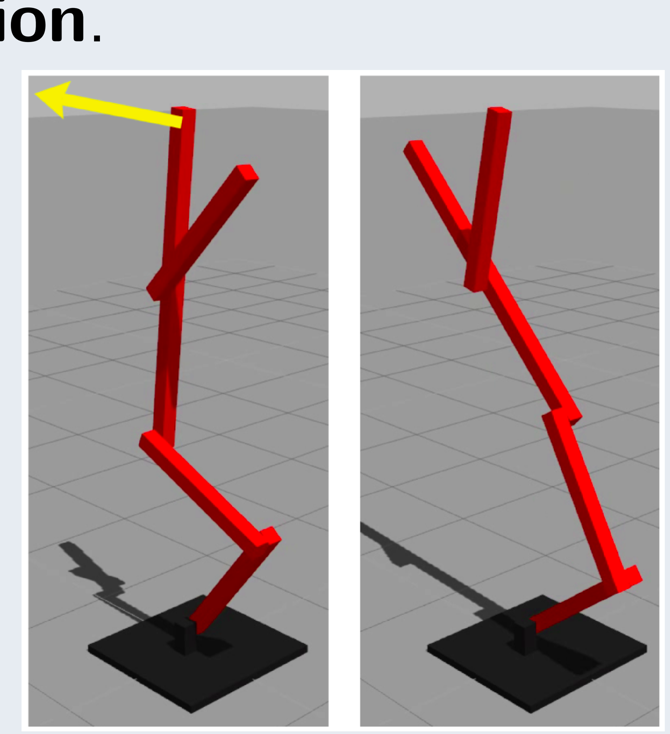


An A.S. relation ensures that the anchor can **track the template with ϵ precision**.



This enables **better push recovery** by allowing centroidal momentum to vary.

Approximate Simulation Relations

Complex anchor system:

$$\Sigma_1 : \begin{cases} \dot{\mathbf{x}}_1 = f_1(\mathbf{x}_1, \mathbf{u}_1) \\ \mathbf{y}_1 = g_2(\mathbf{x}_1) \end{cases}$$

Simpler template system:

$$\Sigma_2 : \begin{cases} \dot{\mathbf{x}}_2 = f_2(\mathbf{x}_2, \mathbf{u}_2) \\ \mathbf{y}_2 = g_2(\mathbf{x}_2) \end{cases}$$

Lyapunov-like Simulation Function:

$$\mathcal{V}(\mathbf{x}_1, \mathbf{x}_2) \geq \|g_1(\mathbf{x}_1) - g_2(\mathbf{x}_2)\|$$

Interface:

$$\mathbf{u}_1 = u_{\mathcal{V}}(\mathbf{u}_2, \mathbf{x}_1, \mathbf{x}_2)$$

For $\gamma(\|\mathbf{u}_2\|) < \mathcal{V}(\mathbf{x}_1, \mathbf{x}_2)$,

$$\frac{\partial \mathcal{V}}{\partial \mathbf{x}_2} f_2(\mathbf{x}_2, \mathbf{u}_2) + \frac{\partial \mathcal{V}}{\partial \mathbf{x}_1} f_1(\mathbf{x}_1, u_{\mathcal{V}}(\mathbf{u}_2, \mathbf{x}_1, \mathbf{x}_2)) < 0.$$

Approximate Simulation for Linear Systems

Difficult to find $\mathcal{V}$, $u_{\mathcal{V}}$ in general, but for linear systems...

$$\mathcal{V}(\mathbf{x}_1, \mathbf{x}_2) = \sqrt{(\mathbf{x}_1 - \mathbf{P}\mathbf{x}_2)^T \mathbf{M} (\mathbf{x}_1 - \mathbf{P}\mathbf{x}_2)}$$

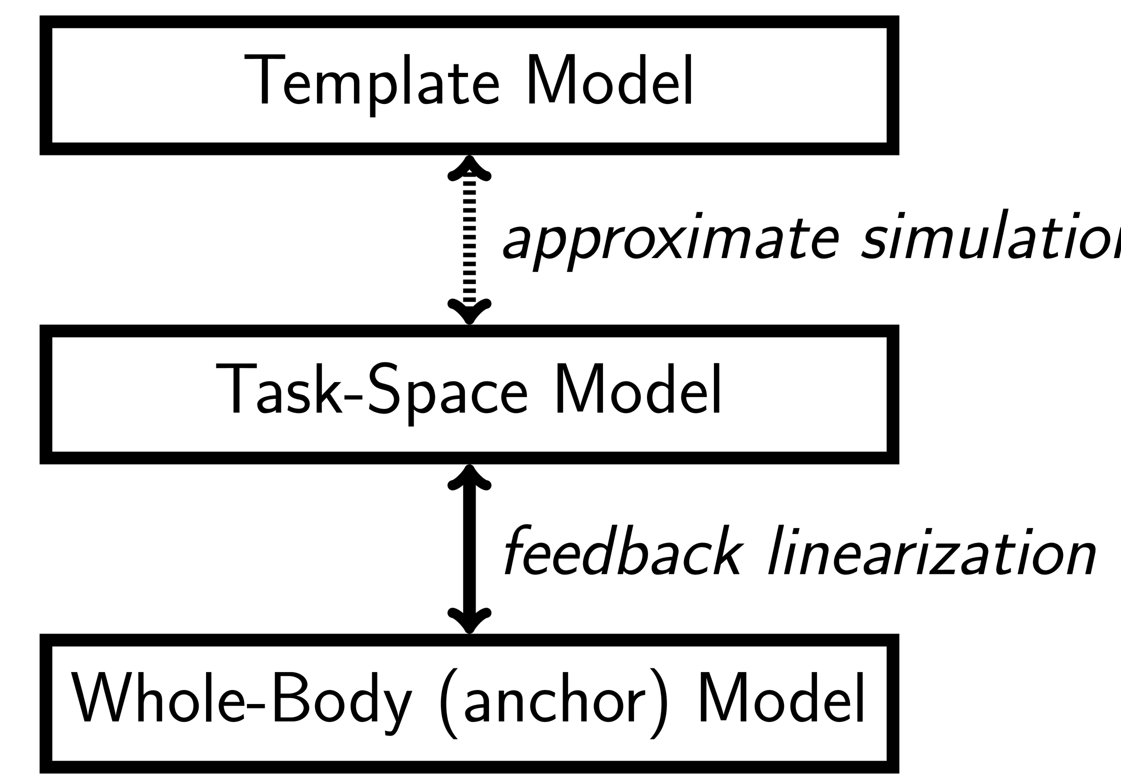
$$u_{\mathcal{V}}(\mathbf{u}_2, \mathbf{x}_1, \mathbf{x}_2) = \mathbf{R}\mathbf{u}_2 + \mathbf{Q}\mathbf{x}_2 + \mathbf{K}(\mathbf{x}_1 - \mathbf{P}\mathbf{x}_2)$$

$$\gamma(\mathbf{u}_2) = \frac{\|\sqrt{\mathbf{M}}(\mathbf{B}_1\mathbf{R} - \mathbf{P}\mathbf{B}_2)\|}{\lambda} \mathbf{u}_2$$

where

- $\mathbf{P}\mathbf{A}_2 = \mathbf{A}_1\mathbf{P} + \mathbf{B}_1\mathbf{Q}$
- $\mathbf{C}_2 = \mathbf{C}_1\mathbf{P}$
- $\mathbf{K}$ is stabilizing feedback gain for Σ_1
- $\mathbf{M}$ certifies convergence of Σ_1 to zero with rate λ under $\mathbf{u}_1 = \mathbf{K}\mathbf{x}_1$

Templates and Anchors



Template: Linear Inverted Pendulum.

$$\dot{\mathbf{x}}_{lip} = \mathbf{A}_{lip}\mathbf{x}_{lip} + \mathbf{B}_{lip}u_{lip}$$

where u_{lip} is the center-of-pressure position.

Task-Space: Centroidal Dynamics.

$$\dot{\mathbf{x}}_{task} = \mathbf{A}_{lip}\mathbf{x}_{task} + \mathbf{B}_{lip}u_{task}$$

where

$$\mathbf{x}_{task} = \begin{bmatrix} \mathbf{p}_G \\ \mathbf{h}_G \end{bmatrix} \qquad \mathbf{u}_{task} = \dot{\mathbf{h}}_G$$

Anchor: Rigid-Body Model.

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \boldsymbol{\tau}_g = \boldsymbol{\tau}$$

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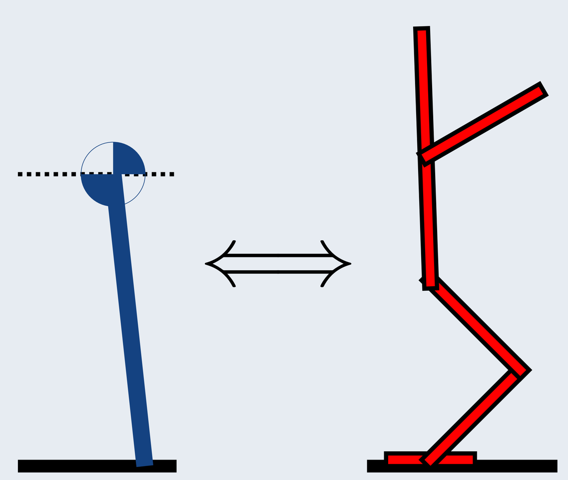

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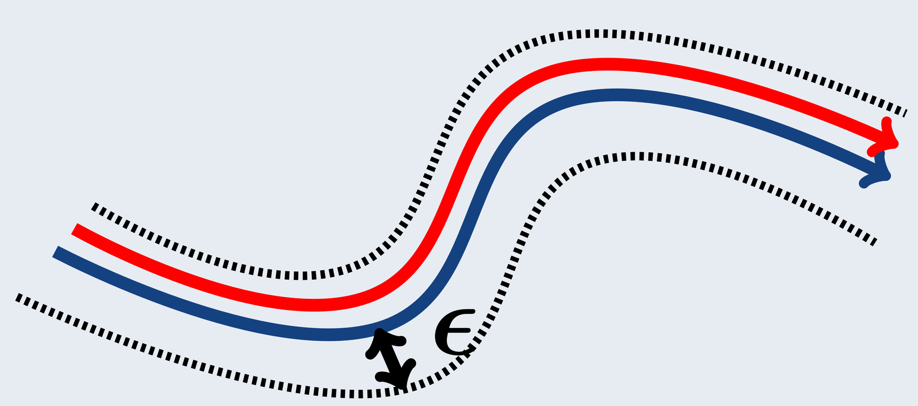

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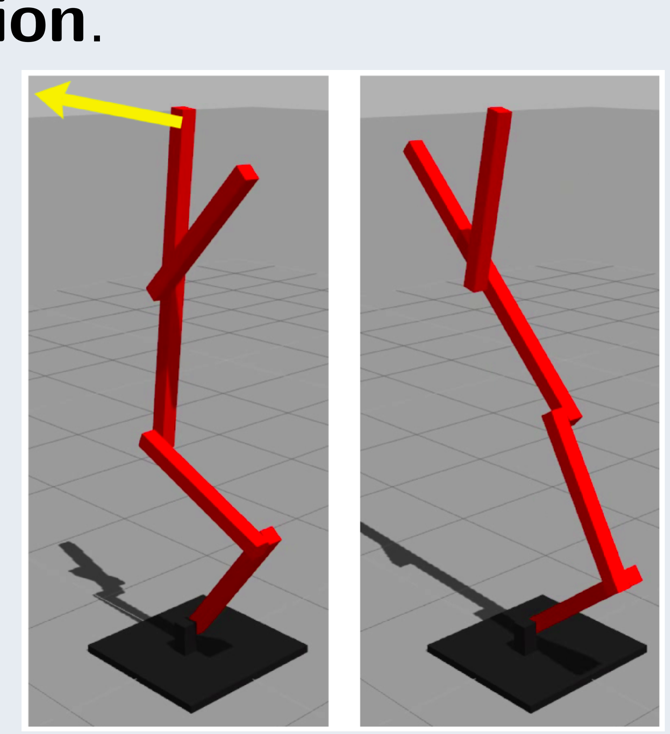

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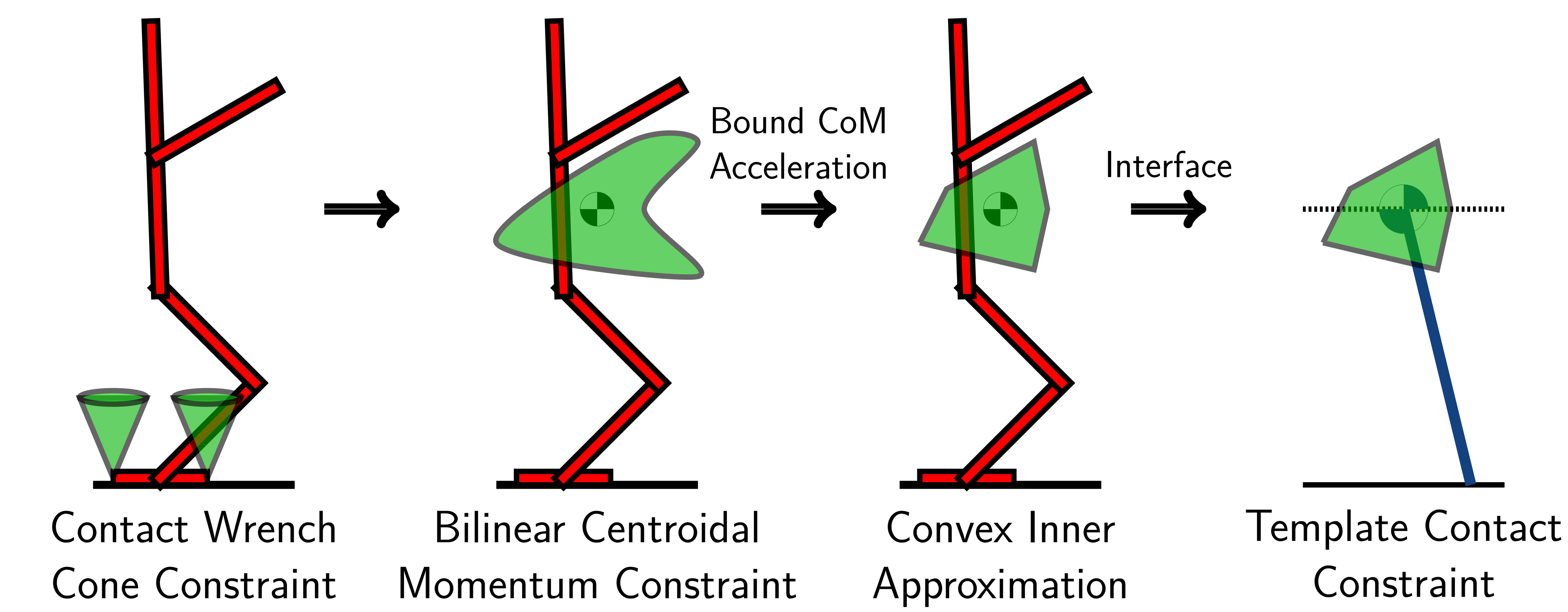
Projecting Contact Constraints to the Template



Constraint Linearization

CWC Constraint on contact force:

$$\mathbf{A}\mathbf{f}_0 \le 0$$

Bilinear constraint on centroidal momentum $\mathbf{h}_G$:

$$\mathbf{A}\,{}^0\mathbf{X}_G^*(\dot{\mathbf{h}}_G - \begin{bmatrix}\mathbf{0}\\ m\mathbf{g}\end{bmatrix}) \le 0$$

Rewrite in terms of linear ($\mathbf{l}_G$) and angular ($\mathbf{k}_G$) momentum:

$$\mathbf{A}\,{}^0\mathbf{X}_G^*\dot{\mathbf{h}}_G \le \mathbf{A}\,{}^0\mathbf{X}_G^*\begin{bmatrix}\mathbf{0}\\ m\mathbf{g}\end{bmatrix}$$

$$\mathbf{A}\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix}\dot{\mathbf{k}}_G + \mathbf{A}\begin{bmatrix}\mathbf{S}(\mathbf{p}_G)\\ \mathbf{I}\end{bmatrix}\dot{\mathbf{l}}_G \le \mathbf{A}\begin{bmatrix}\mathbf{I} & \mathbf{S}(\mathbf{p}_G)\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ m\mathbf{g}\end{bmatrix}$$

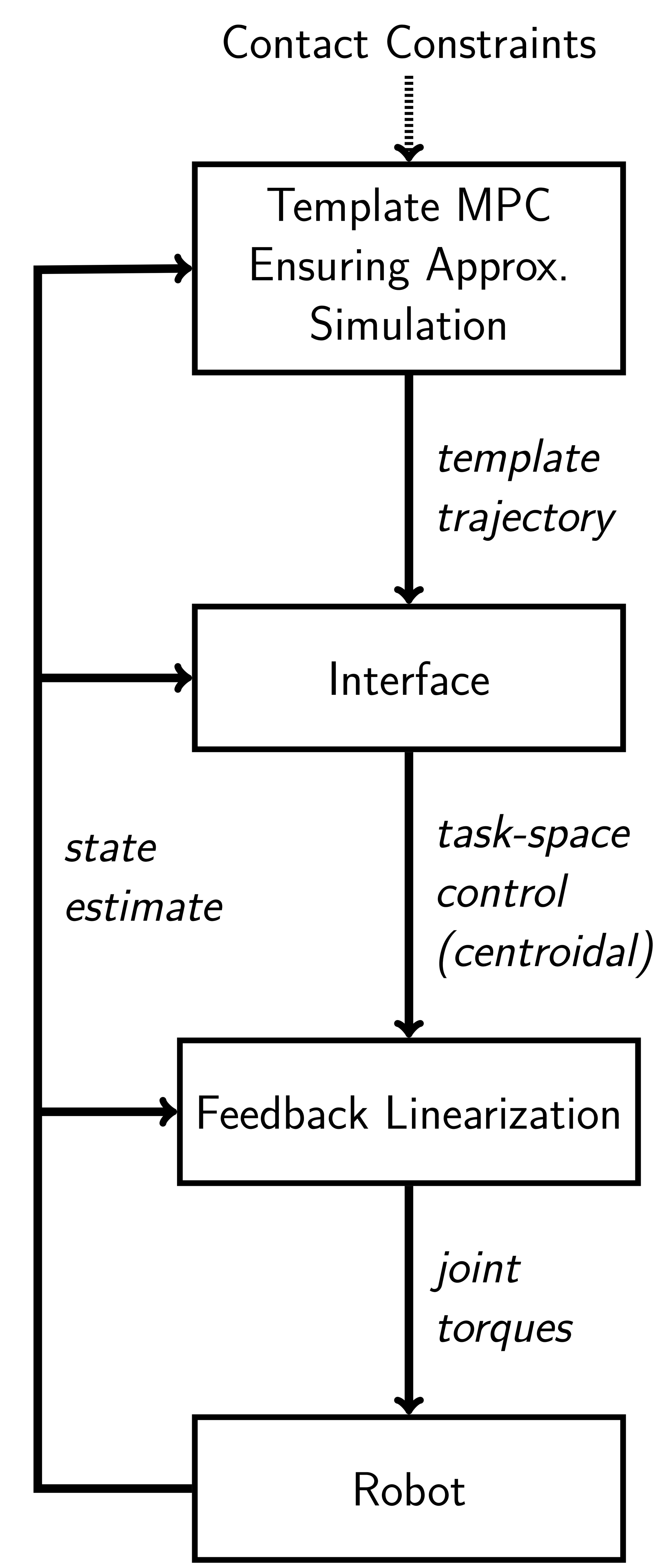
Linear constraint if $\|\dot{\mathbf{l}}_G\|_\infty \le l_{max}$:

$$\mathbf{A}\dot{\mathbf{h}}_G + \mathbf{A}\begin{bmatrix}\mathbf{S}(m\mathbf{g}) - \mathbf{S}(\dot{\mathbf{l}}_G)\\ \mathbf{0}\end{bmatrix}\mathbf{p}_G \le \mathbf{A}\begin{bmatrix}\mathbf{0}\\ m\mathbf{g}\end{bmatrix}$$

Details

- Template MPC accounts for friction
- Interface enforces formal relation with centroidal dynamics
- Feedback linearization determines joint torques, no need to consider friction

Our Control Framework



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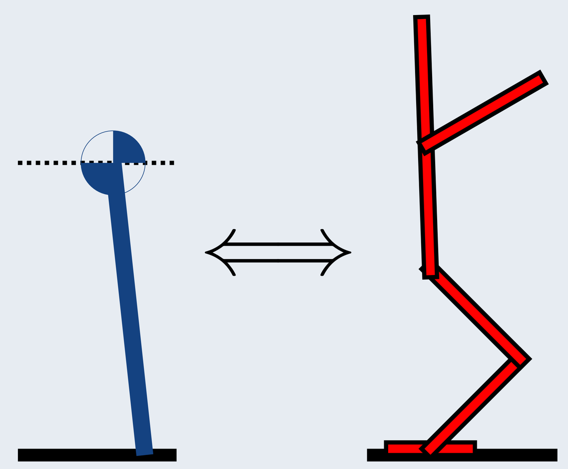

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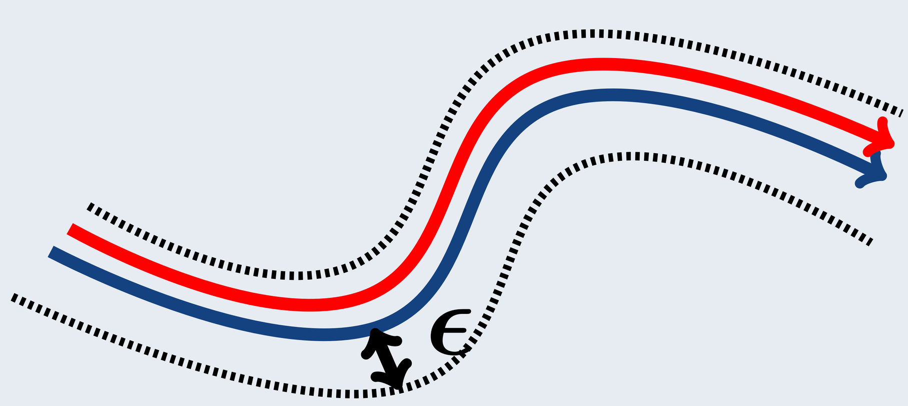

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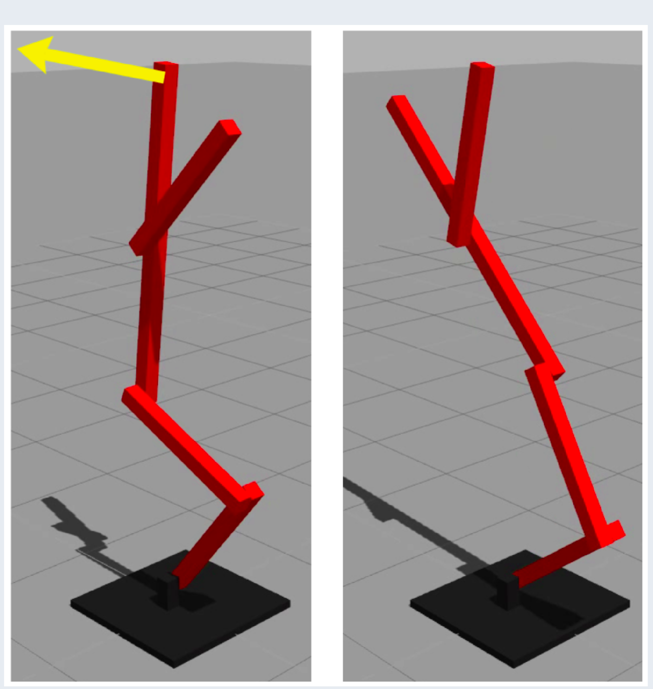

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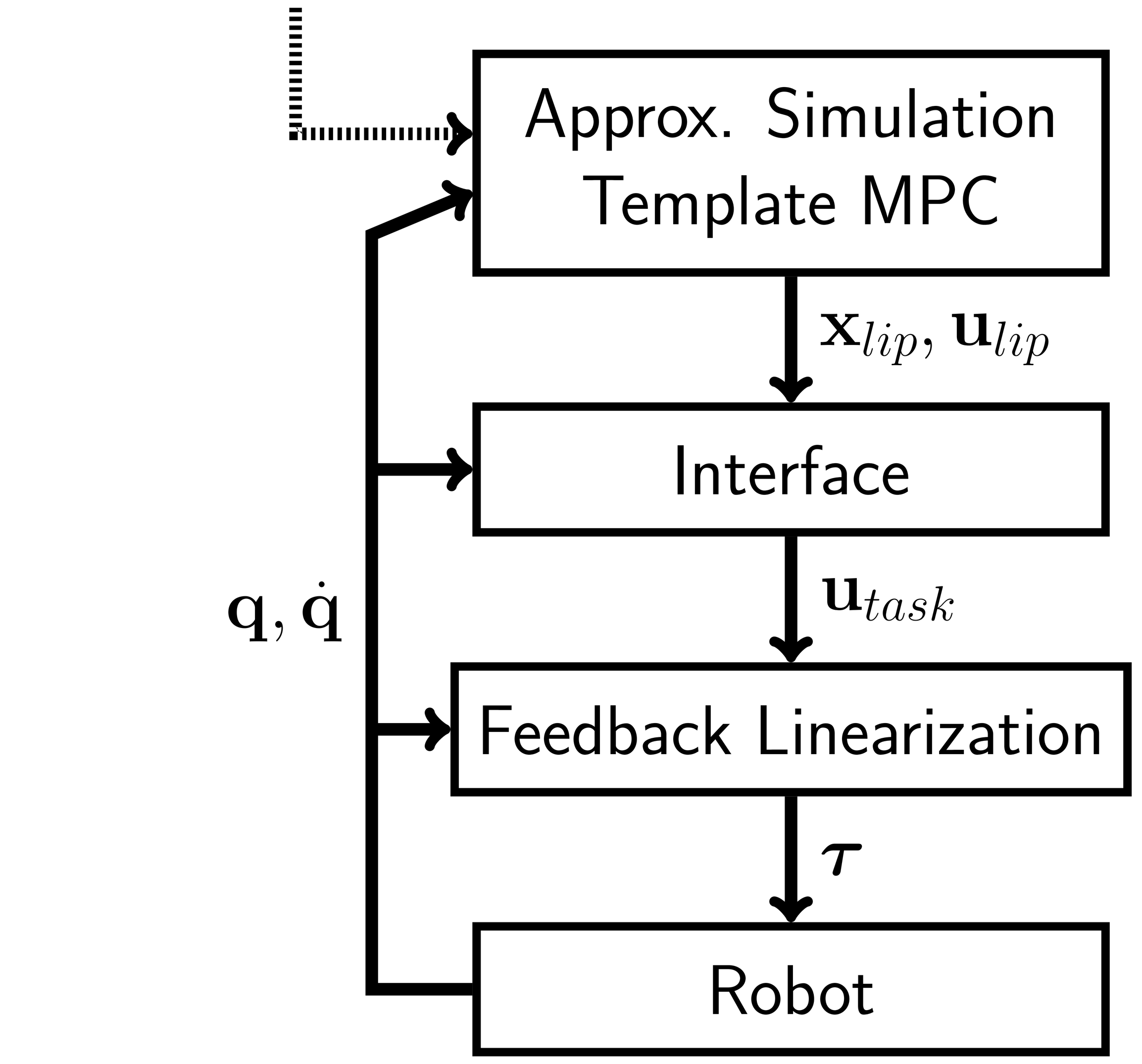
Proposed Control Framework



Contact-aware Template MPC

$$\min \sum_{t=0}^{N-1} L\left(\mathbf{x}_{lip}(t), \mathbf{u}_{lip}(t)\right) + L_f\left(\mathbf{x}_{lip}(N)\right)$$

$$\text{s.t. } \mathbf{x}_{lip}(0), \mathbf{x}_{task}(0) \text{ given}$$

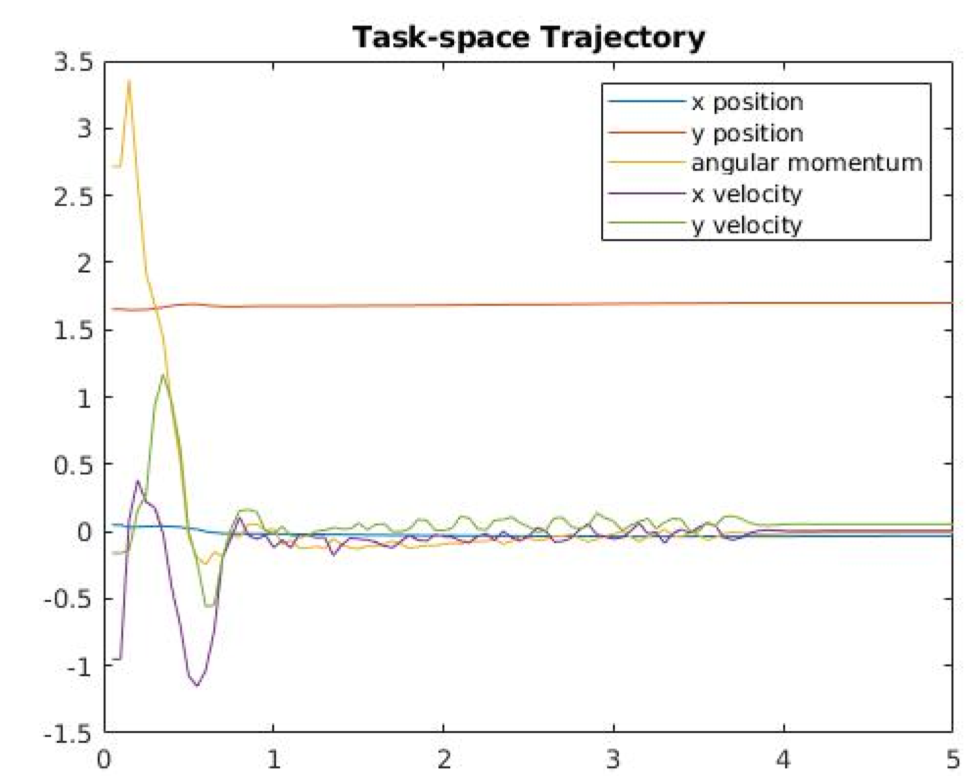
$$\dot{\mathbf{x}}_{lip} = \mathbf{A}_{lip}\mathbf{x}_{lip} + \mathbf{B}_{lip}u_{lip}$$

$$\dot{\mathbf{x}}_{task} = \mathbf{A}_{task}\mathbf{x}_{task} + \mathbf{B}_{task}\mathbf{u}_{task}$$

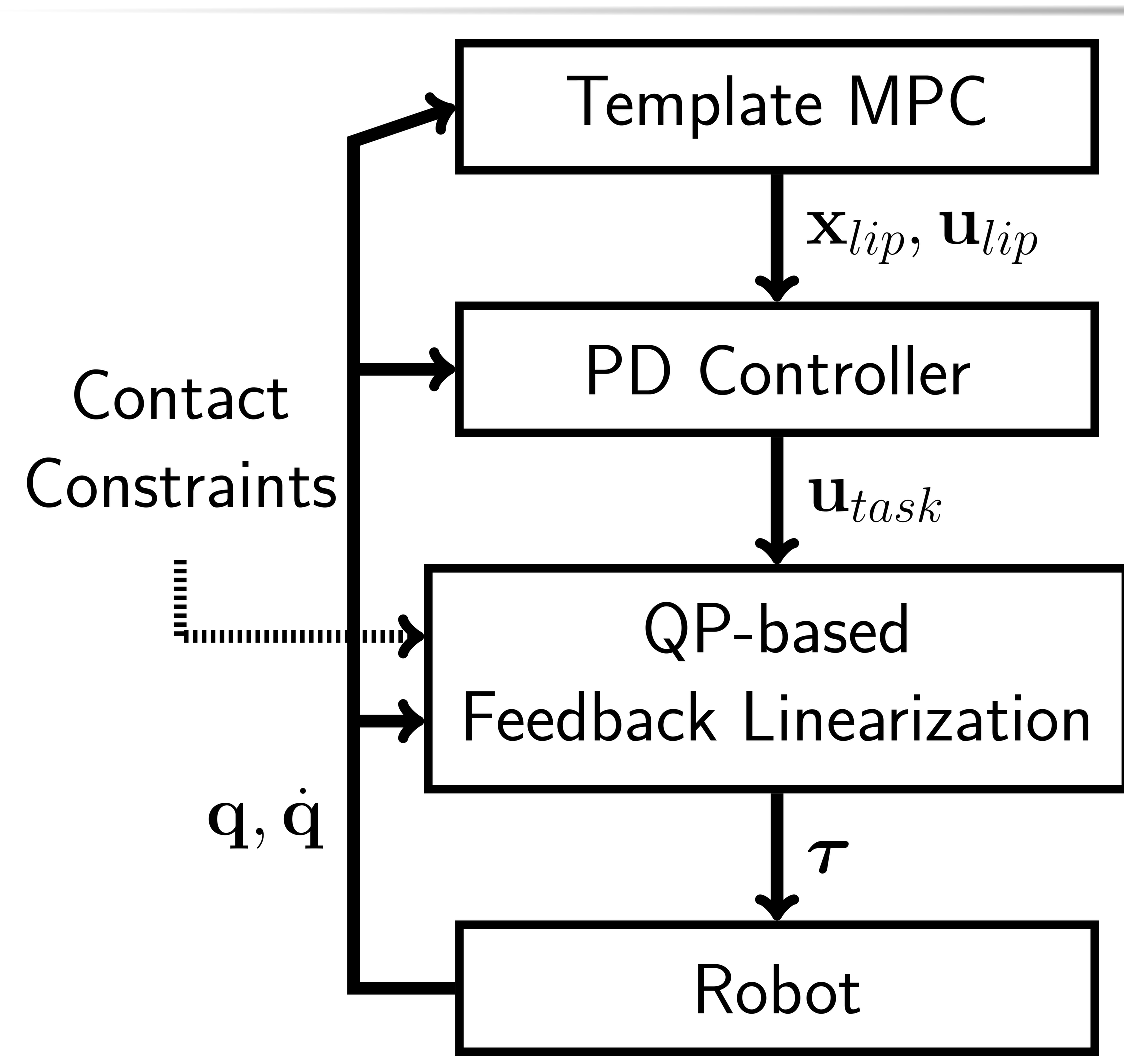
$$\mathbf{u}_{task} = \mathbf{R}u_{lip} + \mathbf{Q}\mathbf{x}_{lip} + \mathbf{K}(\mathbf{x}_{task} - \mathbf{P}\mathbf{x}_{lip})$$

$$\mathbf{A}_{cwc}\begin{bmatrix}\mathbf{x}_{task} \\ \mathbf{u}_{task}\end{bmatrix} \leq \mathbf{b}_{cwc}$$

$$\|\dot{\mathbf{l}}_G\|_\infty \leq \dot{l}_{max}$$



Standard Control Framework

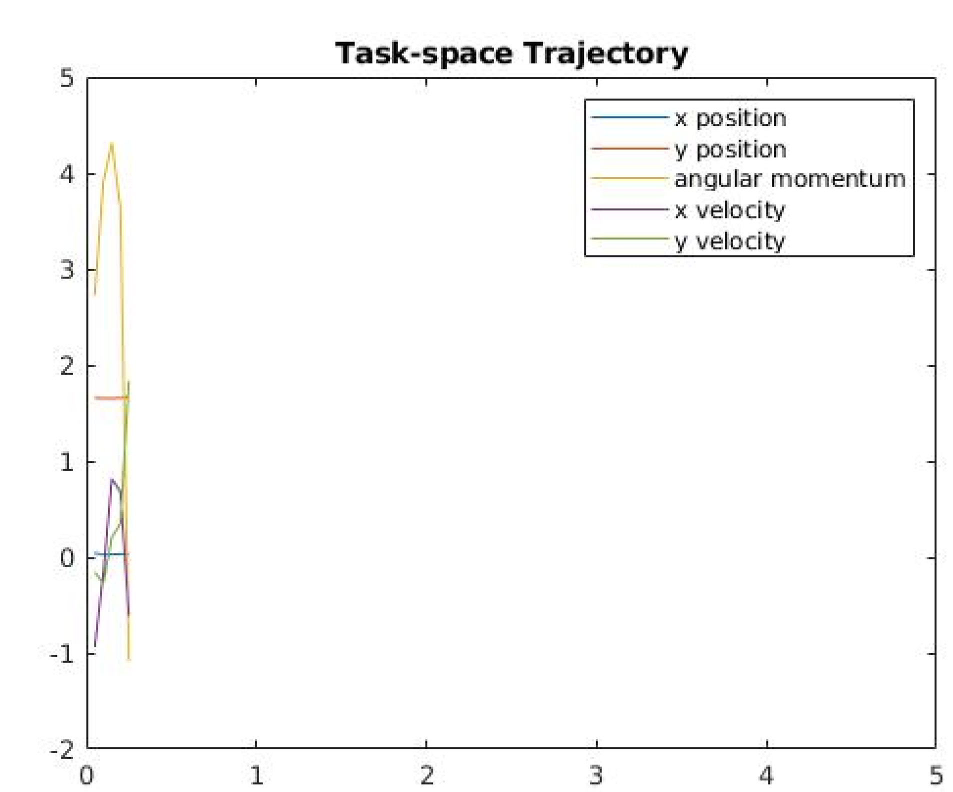


Traditional Template MPC

$$\min \sum_{t=0}^{N-1} L\left(\mathbf{x}_{lip}(t), \mathbf{u}_{lip}(t)\right) + L_f\left(\mathbf{x}_{lip}(N)\right)$$

$$\text{s.t. } \mathbf{x}_{lip}(0), \mathbf{x}_{task}(0) \text{ given}$$

$$\dot{\mathbf{x}}_{lip} = \mathbf{A}_{lip}\mathbf{x}_{lip} + \mathbf{B}_{lip}u_{lip}$$



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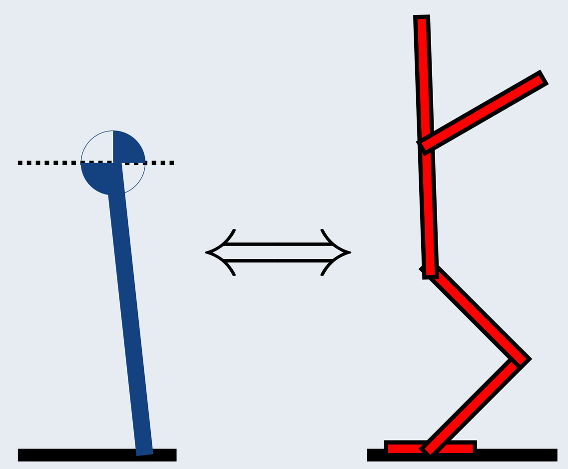

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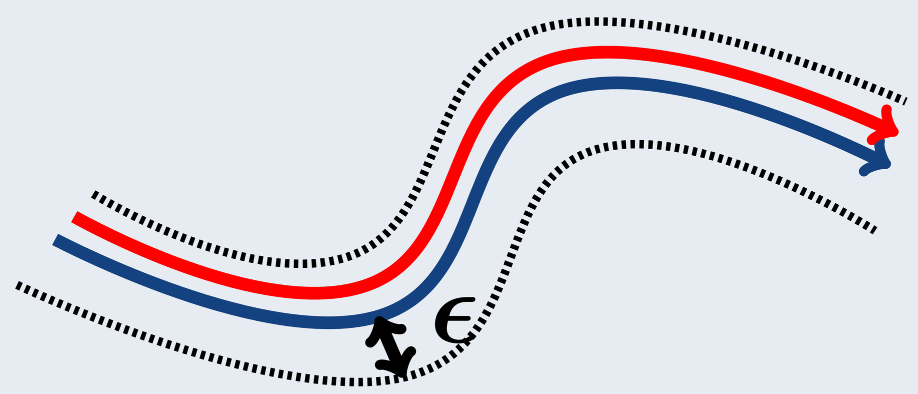

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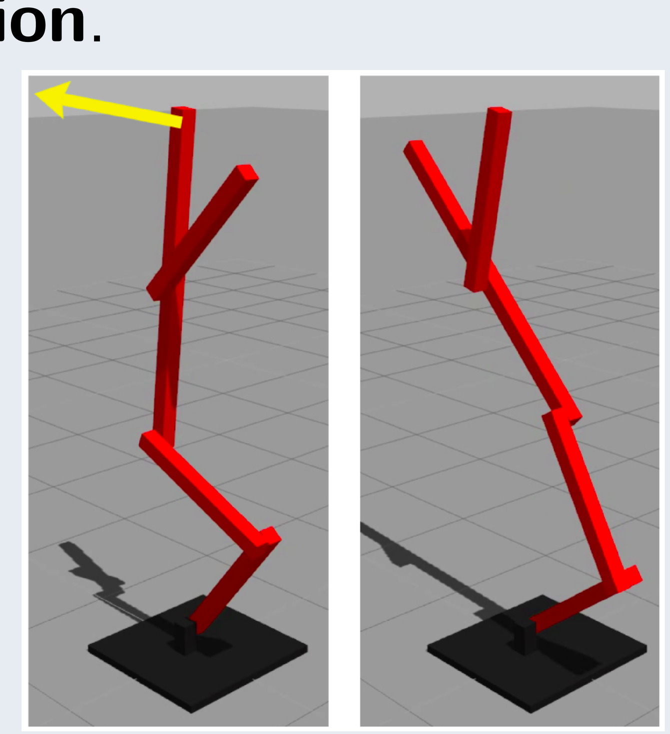

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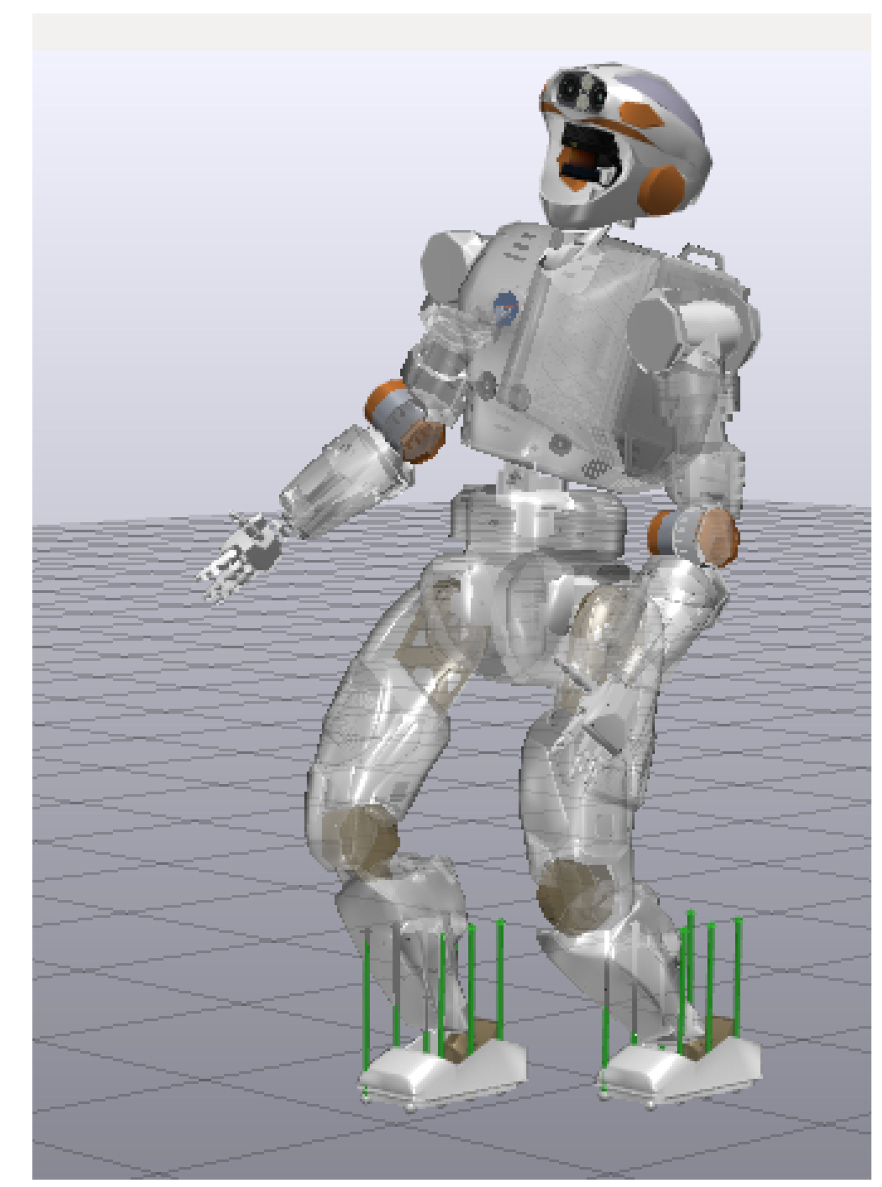
Future Work: Walking Control

Challenges:

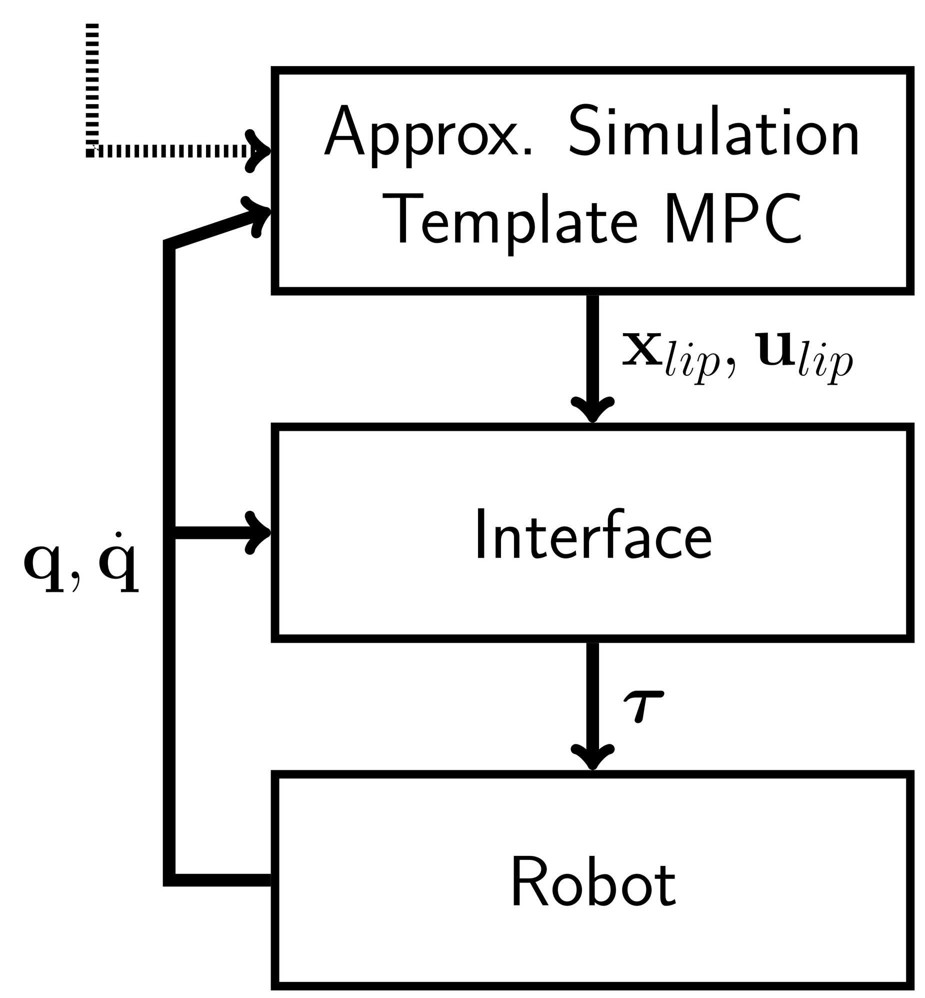
- Impact disturbances: robust approximate simulation
- Multiple support

Anticipated Benefits:

- Template ensures contact feasibility
- Less parameter tuning
- More robust



Future Work: Beyond Task-Space Approximate Simulation

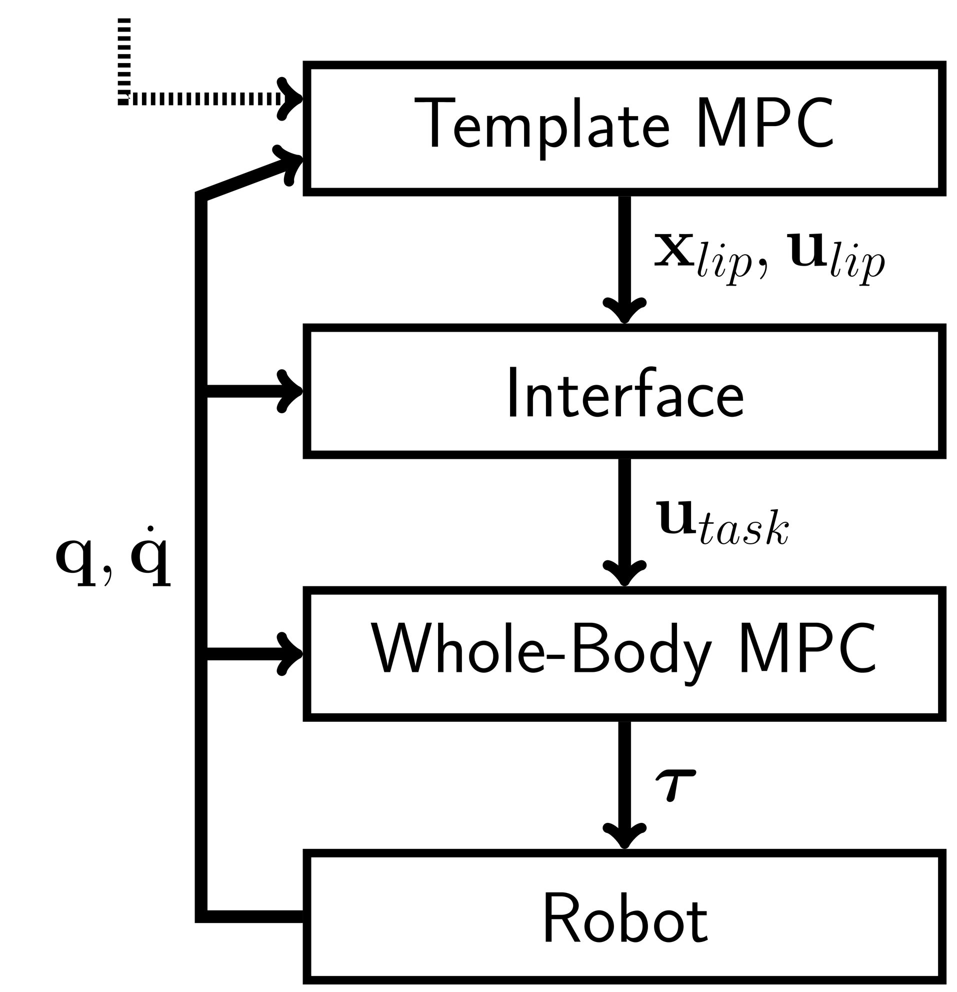


Goal: establish formal connection with whole-body dynamics directly

- Use SOS techniques to find $\mathcal{V}(\mathbf{x}_1, \mathbf{x}_2)$
- D-SOS/SD-SOS to address scalability challenges [1]
- Interface = closed-form $(\mathbf{x}_{lip}, \mathbf{u}_{lip}) \mapsto \boldsymbol{\tau}$

Future Work: Enabling Whole-Body MPC

DDP is promising, but friction constraints are difficult to handle [3].



Use approximate simulation to enforce contact constraints with the template.

References

[1] A. A. Ahmadi and A. Majumdar. *SIAM Journal on Applied Algebra and Geometry*, 3(2):193–230, 2019.

[2] A. Girard and G. J. Pappas. *Automatica*, 45(2):566–571, 2009.

[3] Y. Tassa, T. Erez, and E. Todorov. In *2012 IEEE/RSJ International Conference on Intelligent Robots and Systems*, pages 4906–4913. IEEE, 2012.

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